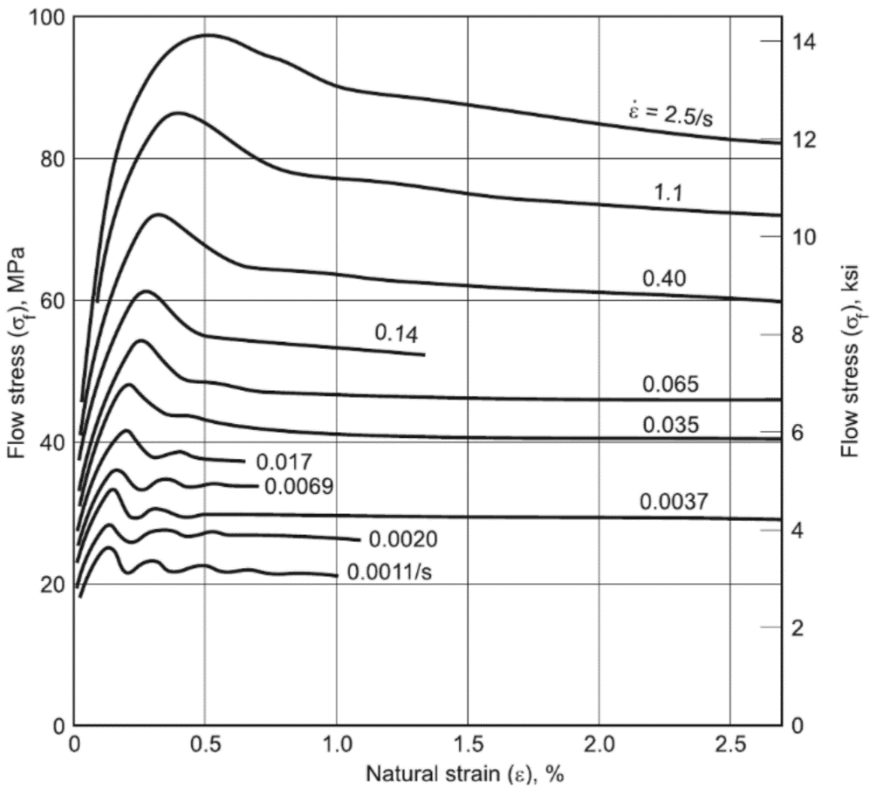
Plasticity and Deformation Processes

Yielding criteria

Deformation and plastic behavior of metals

Effect of strain rate at high T

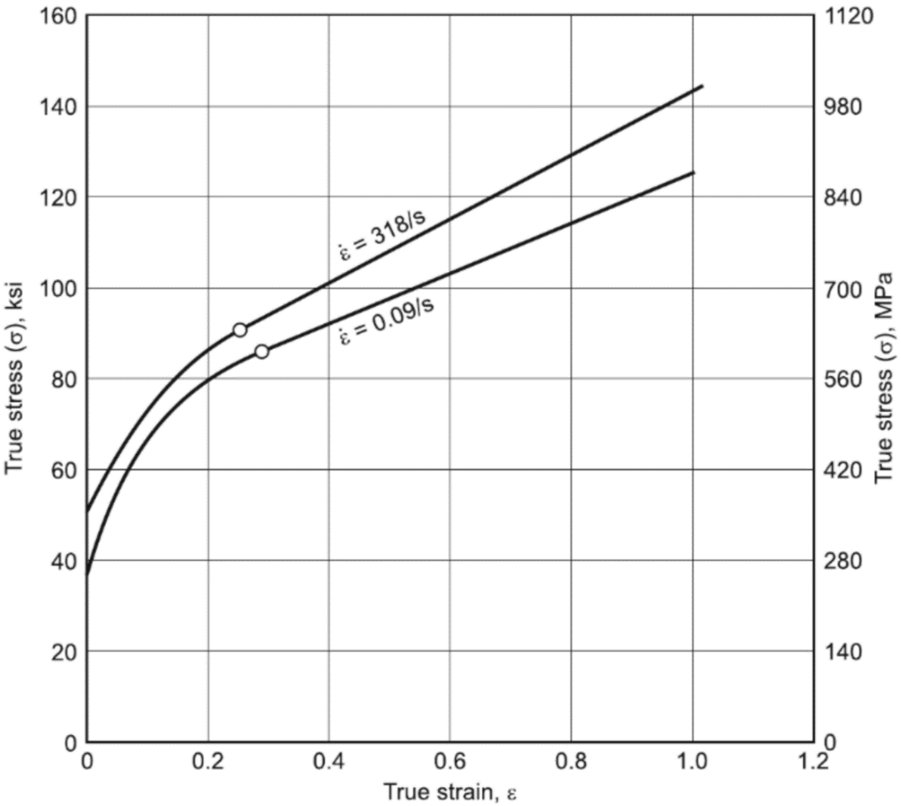


CS.032 1025 carbon (0.25% C) steel, flow stress-strain curves at various strain rates

Temperature (T) = 1100 °C (2012 °F). Stress-strain curves show that at higher strains the flow stress is approximately constant. This is increasingly true at smaller strain rates ($\dot{\varepsilon}$). Curves were obtained in hot torsion experiments. UNS G10250

Source: K. Lange, Ed., *Handbook of Metal Forming,* McGraw-Hill, 1985, p 16.11

Effect of strain rate at low T

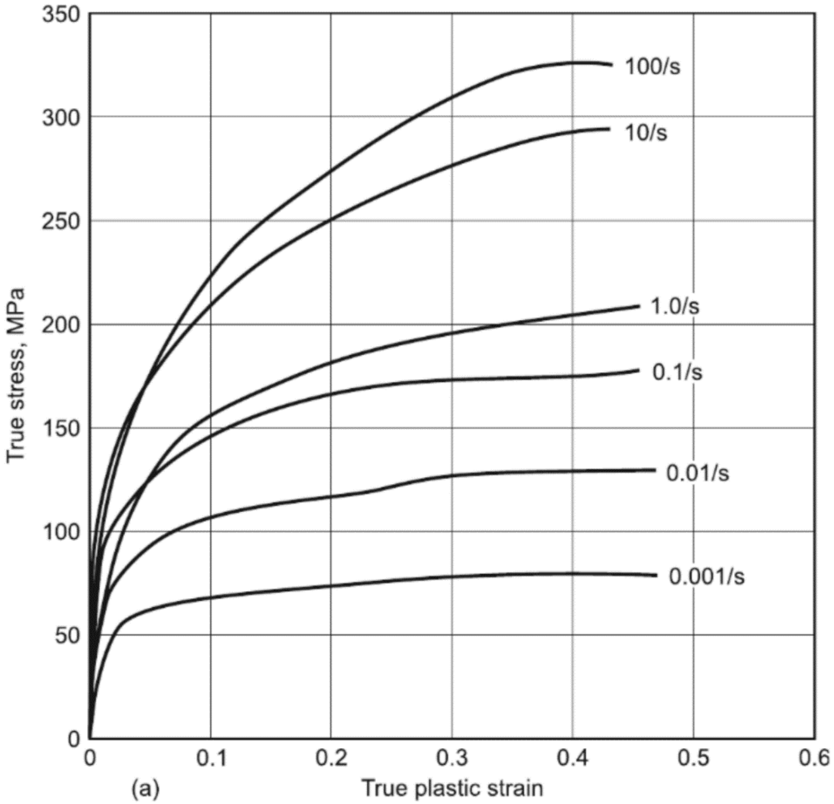


CS.038 1112 carbon steel, true stress-strain curves with effect of strain rate

True stress-strain curves for 1112 steel at different strain rates at 21 °C (70 °F). When metals are tested in tension at different strain rates, the flow stress corresponding to a given strain is found to increase with strain rate. The following equation is frequently used to relate flow stress and strain rate at a given strain and temperature: $\sigma = \sigma_1 \dot{\varepsilon}^m$, where $\dot{\varepsilon} = d\varepsilon/dt$ and σ_1 and m are material constants. The exponent m (strain-rate sensitivity) is found to increase with temperature, especially above the strain recrystallization temperature. In the hot-working region, metals tend to approach the behavior of a Newtonian liquid for which $m = 1$.

Source: M.C. Shaw, *Metal Cutting Principles*, Clarendon Press, Oxford, 1984, p 69

Effect of strain rate at high T

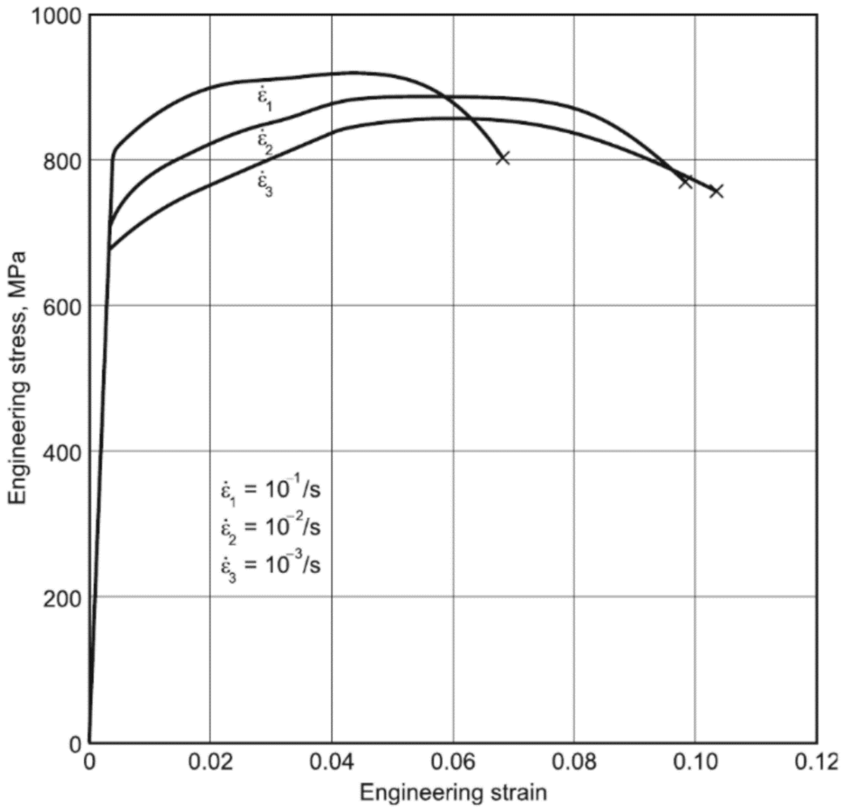


HS.005 Microalloyed high-strength low-alloy (HSLA) steel, compressive true stress-true plastic strain curves at different strain rates

Hot rolled. Thermomechanical processing typically includes rough rolling, 1100–1240 °C (2012–2264 °F), and finish rolling, 810–900 °C (1490–1652 °F), fast cooling to 700 °C (1292 °F), and air cooling. (a) Tested at 900 °C. (b) At 1200 °C. Composition: Fe-0.08C-1.3Mn-0.3Si-0.2Ni-0.08V-0.05Nb-0.015P-0.008S

Source: N.S. Mishra, in *Hot Working Guide A Compendium of Processing Maps*, Y.V.R.K Prasad and S. Sasidhara, Ed., ASM International, 1997, p 337

Effect of strain rate on yield strength

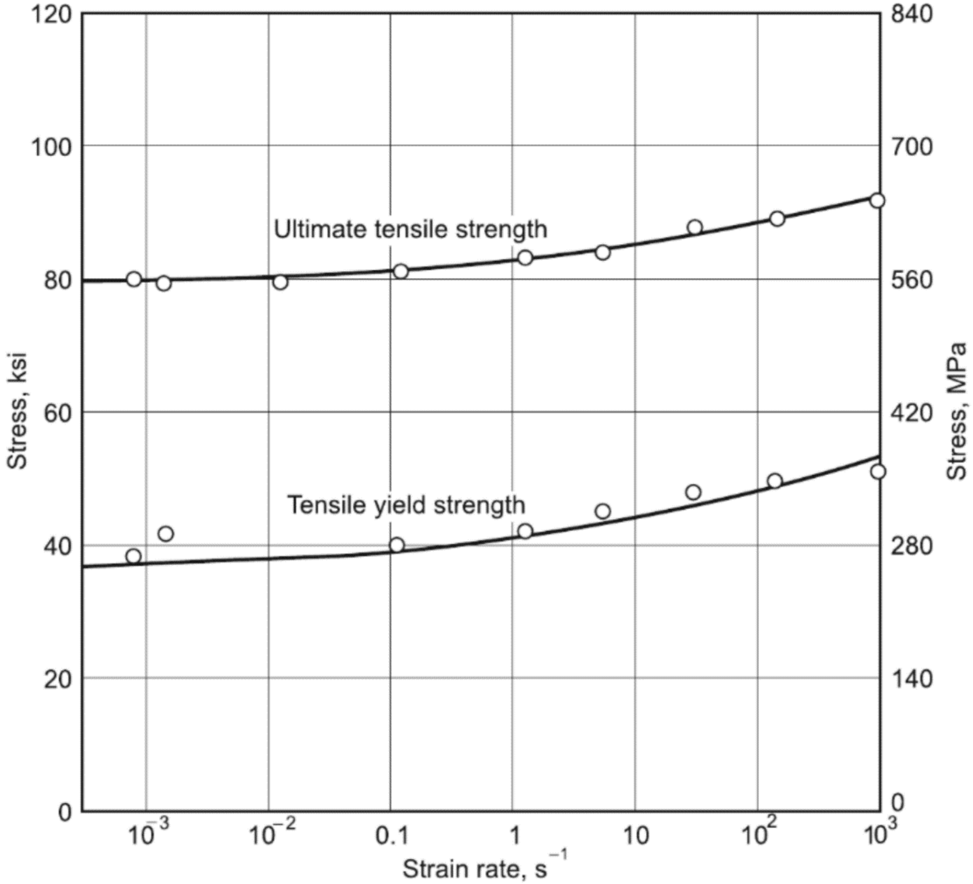


CS.033 1040 carbon steel, engineering stress-strain curves with effect of strain rate

Effect of different strain rates on the tensile response. The yield stress and flow stresses at different values of strain increase with strain rate. The work-hardening rate (m), on the other hand, is not as sensitive to strain rate. This illustrates the importance of correctly specifying the strain rate when giving the yield stress of a metal. Not all metals exhibit a high strain-rate sensitivity. Aluminum and some of its alloys have either 0 or $-m$. In general, m varies between 0.02 and 0.2 for homologous temperatures between 0 and 0.9 (90% of melting point in K). Therefore, one would have, at the most, an increase of 15% in the yield stress by doubling the strain rate. UNS G10400

Source: M.A. Meyers and K.K. Chawla, *Mechanical Metallurgy: Principles and Applications,* Prentice-Hall, 1984, p 572

Effect of strain rate on yield strength, UTS



SS.055 310 annealed stainless steel sheet, effect of strain rate on mechanical properties

Sheet thickness = 1.60 mm (0.063 in.). Composition: Fe-25Cr-20.5Ni. UNS S31000

Source: R.G. Davies and C.L. Magee, The Effect of Strain-Rate upon the Tensile Deformation of Metals, *J. Eng. Mater. Technol.*, April 1975, p 151. As published in *Aerospace Structural Metals Handbook,* Vol 2, Code 1305, CINDAS/USAF CRDA Handbooks Operation, Purdue University, 1995, p 22

Understanding plastic deformation of metals is necessary for controlling deformation processes

We can explain the plasticity of metals using a plasticity theory that consists of

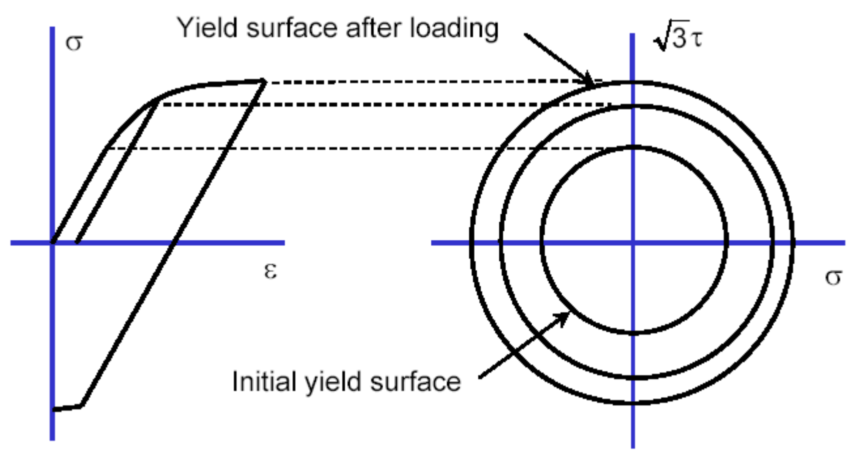
1. A yield function to determine when plastic flow initiates
2. A flow model which relates the applied stress increments to the resulting plastic strain increments

For example deformation theory relates applied stresses to strains by a dynamic modulus using Hooke's law:

$$\sigma = E_{sec}\varepsilon$$
$$d\sigma = E_{sec}d\varepsilon$$

3. A hardening model that describes the change in the yield criterion as a function of plastic strains

For example the isotropic hardening model assumes that strain hardening corresponds to an enlargement of the yield surface (an increase in yield stress) without change of shape or position of the surface



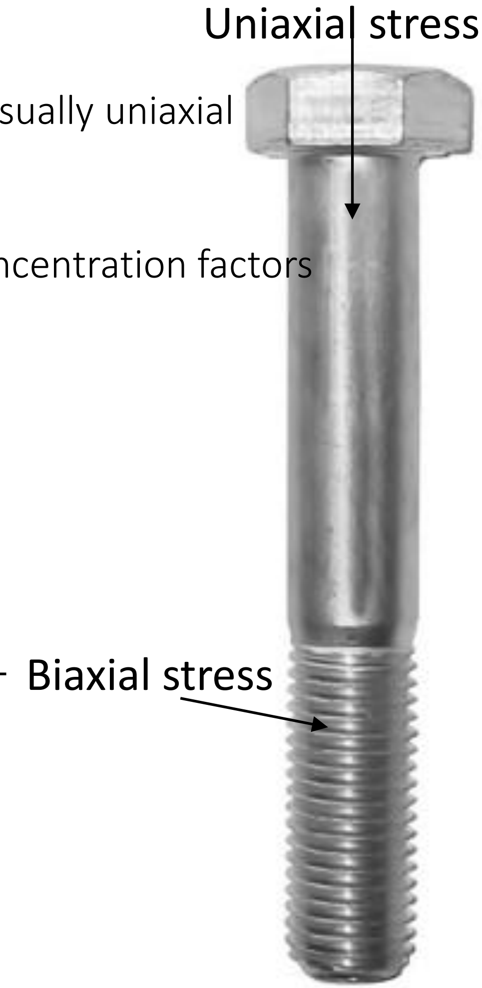
Multiaxial states of stress occur in all types of loadings and so do multiaxial strains

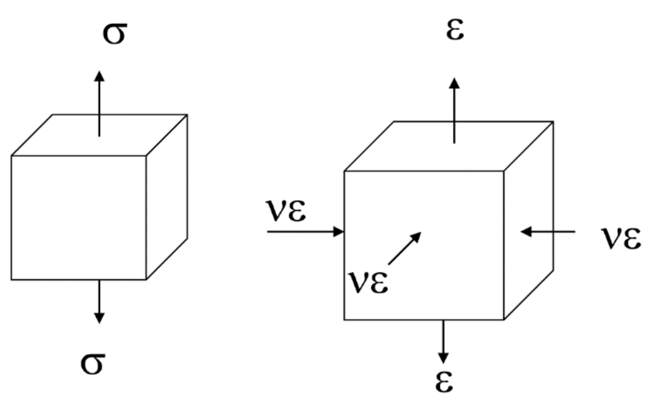
For example the strains in a tensile bar become triaxial

The state of stress around cracks is usually multiaxial and may differ from the state of stress in the bulk of the material

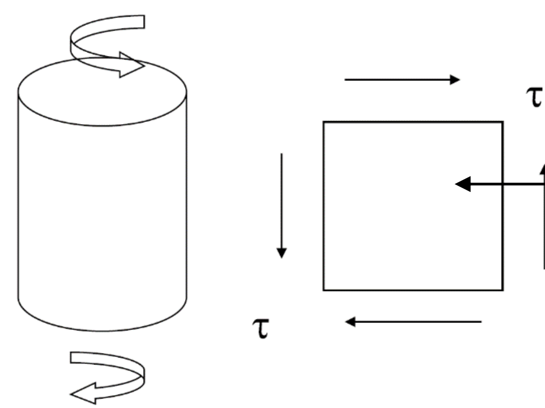
For example the state of stress at the root of a thread is biaxial but it is usually uniaxial in the body of the bolt

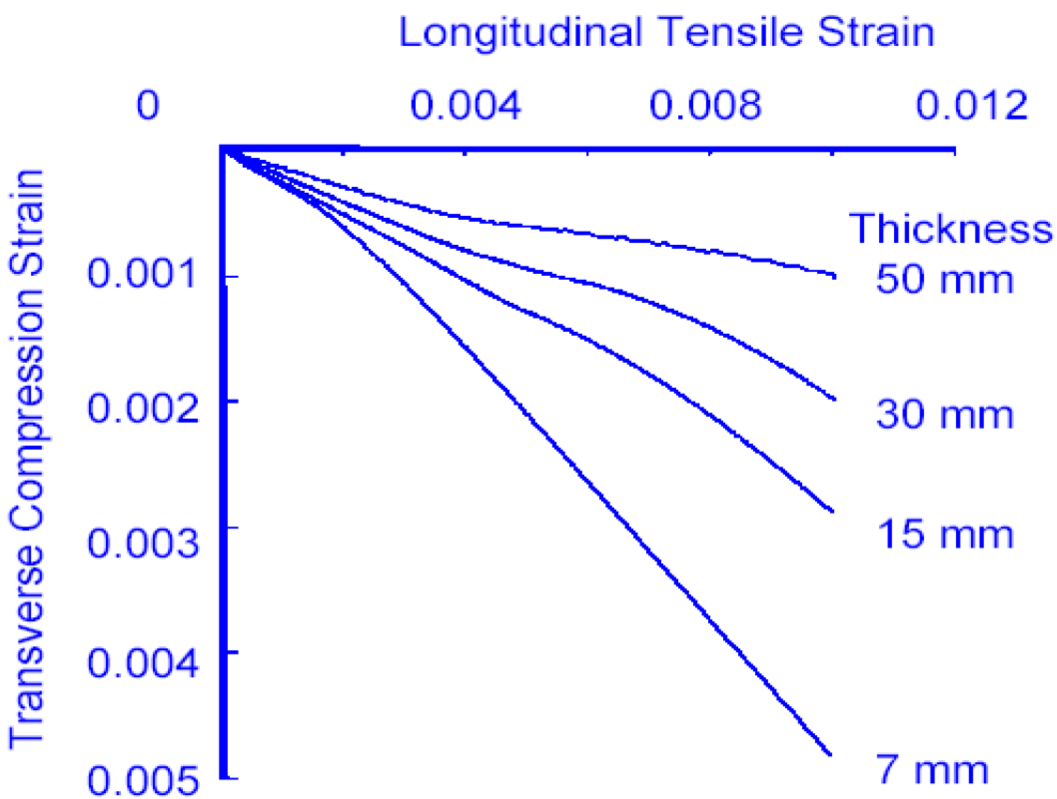
Plastic deformation occurs easier under shear stresses and the stress concentration factors change with stress state

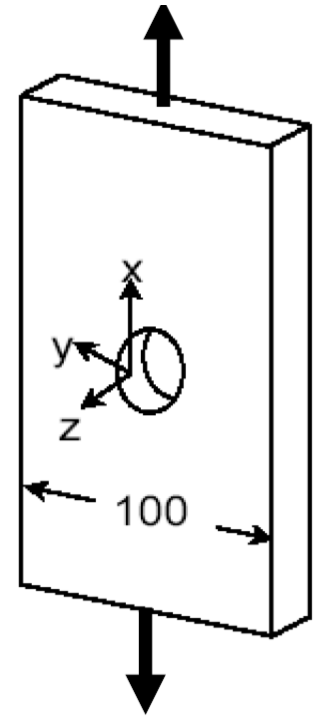




Multiaxial strain







t	ε_x	ε_z	σ_x	σ_z
7	0.01	-0.005	63.5	0
15	0.01	-0.003	70.6	14.1
30	0.01	-0.002	73.0	21.8
50	0.01	-0.001	75.1	29.3

Stress state around a crack in a material under uniaxial tension

We can apply our knowledge obtained from experimental data on the behavior of materials under uniaxial stress (as in a tension test), to multiaxial stress conditions

For example the yield stress obtained from a stress-strain diagram can be used as a yield criterion for materials under multiaxial stress conditions

Multiaxial stress condition may be produced by variation of the position within the bulk of the material (there are only normal stress and strain at the surface perpendicular to the applied normal stress)

It may also be produced from multiaxial stresses

Remember that the state od stress and strain at a point in the material can be identified by 6 stress components and 6 strain components acting on the x, y, z planes

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

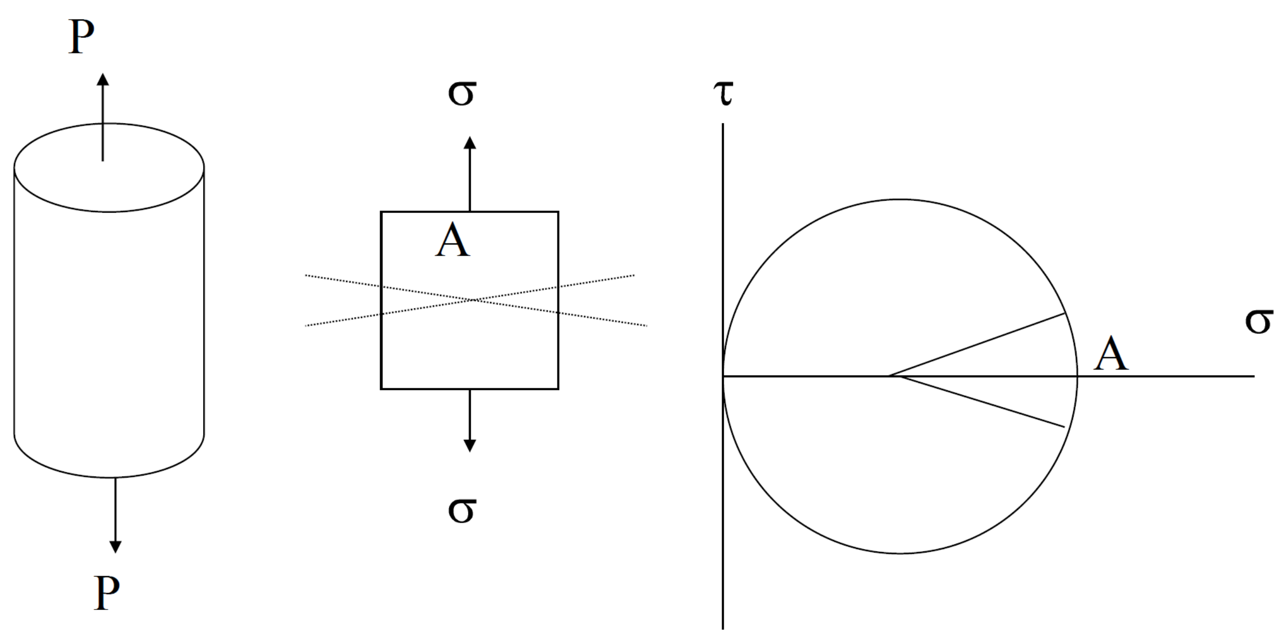
$$\epsilon = \begin{bmatrix} +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} & \frac{\tau_{xy}}{G} & \frac{\tau_{xz}}{G} \\ \frac{\tau_{xy}}{G} & -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} & \frac{\tau_{yz}}{G} \\ \frac{\tau_{xz}}{G} & \frac{\tau_{yz}}{G} & -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \end{bmatrix}$$

We can find stresses and strains acting in any other direction or plane by using transformation equations or graphically using Mohr’s circle

The following parameters are of importance when analyzing the yield criteria at a certain position in the bulk material:

- Maximum normal principal stress, σ_1
- Minimum normal principal stress, σ_2
- Maximum shear stress, τ_{max}
- Effective stress, σ_{eff}
- Uniaxial yield strength, σ_y
- Uniaxial ultimate strength, σ_u

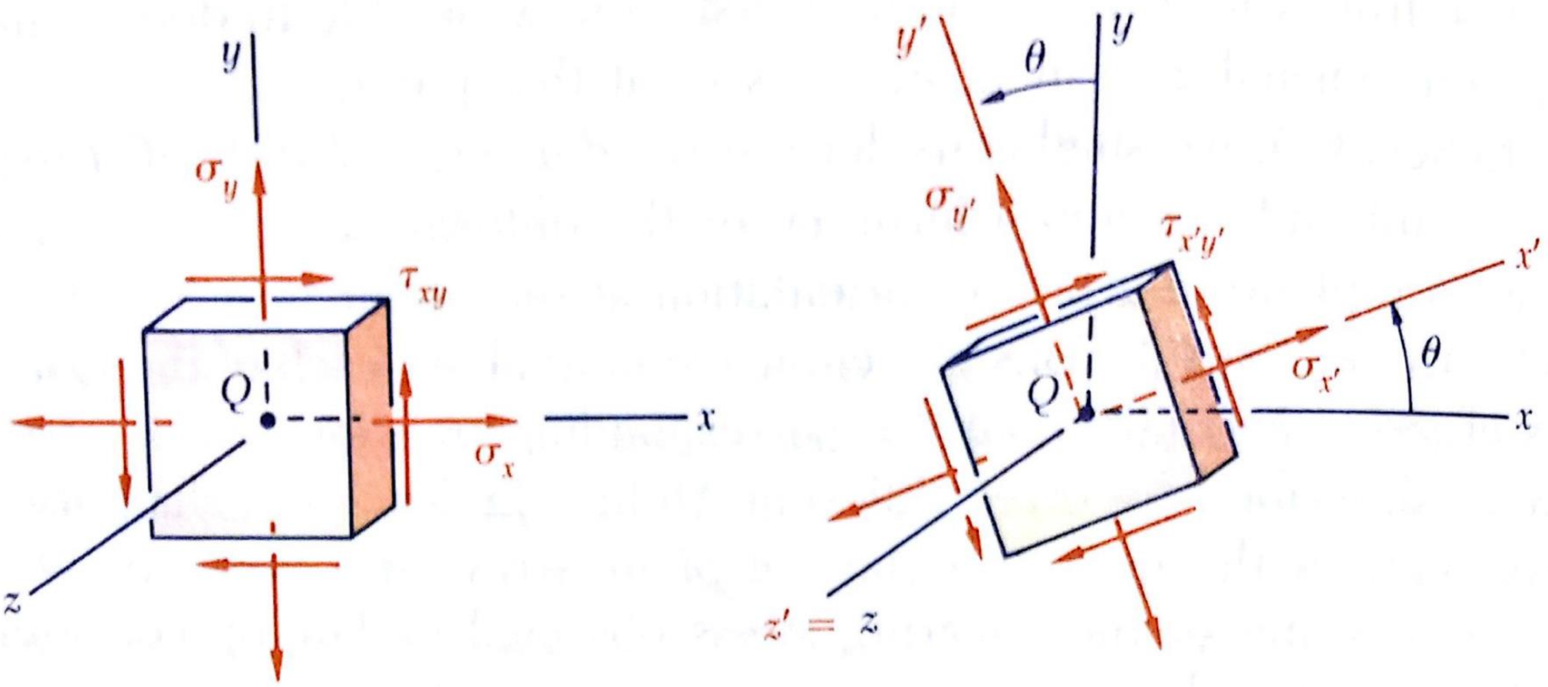
Only a few planes experience the maximum normal stress and the maximum shearing stress
However many other planes experience a very large percentage of these quantities



Yielding depends on the magnitude of the normal and shear stresses applied to a material and also on the local stresses generated at some plane (slip-plane) within the material

Consider a planar material stressed in two directions

A state of plane stress exists at a point Q with $\sigma_z = \tau_{zx} = \tau_{zy} = 0$. The state of plane stress is defined by the stress components $\sigma_x, \sigma_y, \tau_{xy}$ associated with the material shown:



If the material is rotated through an angle θ about the z axis, the stress components change to $\sigma_x{}', \sigma_y{}', \tau_{xy}{}'$ which can be expressed in terms of $\sigma_x, \sigma_y, \tau_{xy}$ and θ

Consider a prismatic element with faces respectively perpendicular to the x, y and x' axes:

If the area of the oblique face is ΔA, the areas of the vertical and horizontal faces are equal to ΔA cosθ, and ΔA sinθ respectively.

The mechanical equilibrium along the x' and y' axes require that

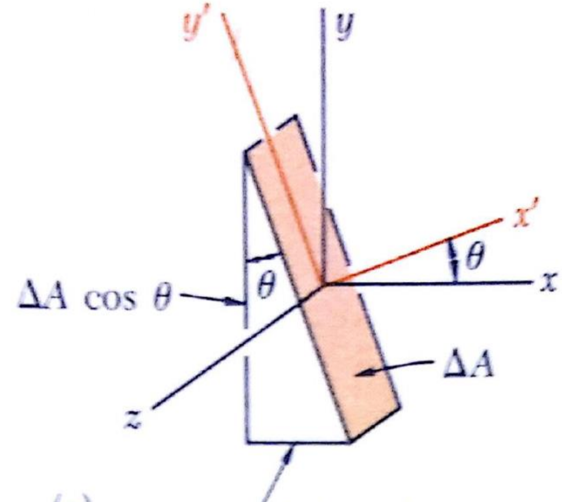
$$\sum F_{x'} = 0,\ \sigma_x{}'\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta - \sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta = 0$$

$$\sum F_{y'} = 0,\ \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta - \sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta = 0$$

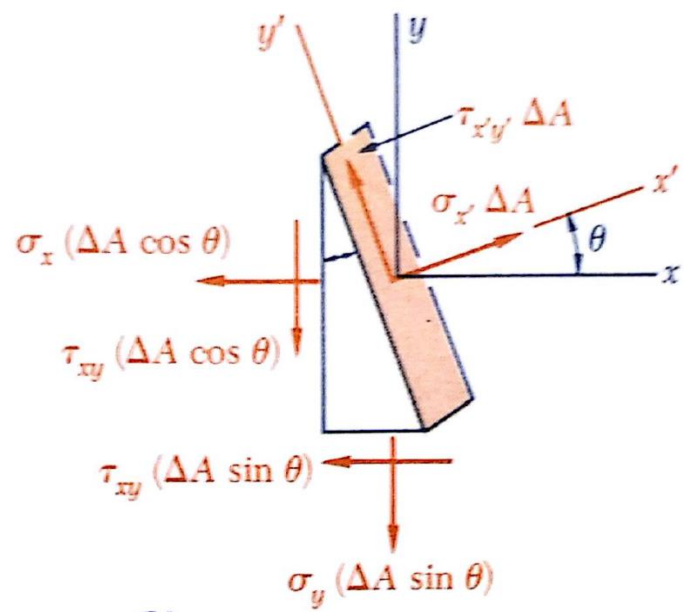
The first equation is solved for $\sigma_x{}'$ and the second for $\tau_{x'y'}$ as

$$\sigma_x{}' = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$



(a)



(b)

After simplifications using trigonometric substitutions we obtain the normal and shear stresses on the rotated material as

$$\sigma_x{}' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y{}' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

The expression for the normal stress $\sigma_y{}'$ is obtained by replacing θ by the angle θ+90 that the y' axis forms with the x axis.

Adding the two normal stresses we see that

$$\sigma_x{}' + \sigma_y{}' = \sigma_x + \sigma_y$$

In the case of plane stress, the sum of the normal stresses exerted on a cubic material is independent of the orientation of the material since $\sigma_z = \sigma_{z'} = 0$

The equations obtained for the normal and shear stresses in the rotated material under plane stress condition are the parametric equations of a circle

If we plot a point M in the rectangular axes with the coordinates $(\sigma_x{}', \tau_{x'y'})$ for any given value of the parameter θ, all the other possible points will lie on a circle.

$$\sigma_x{}' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

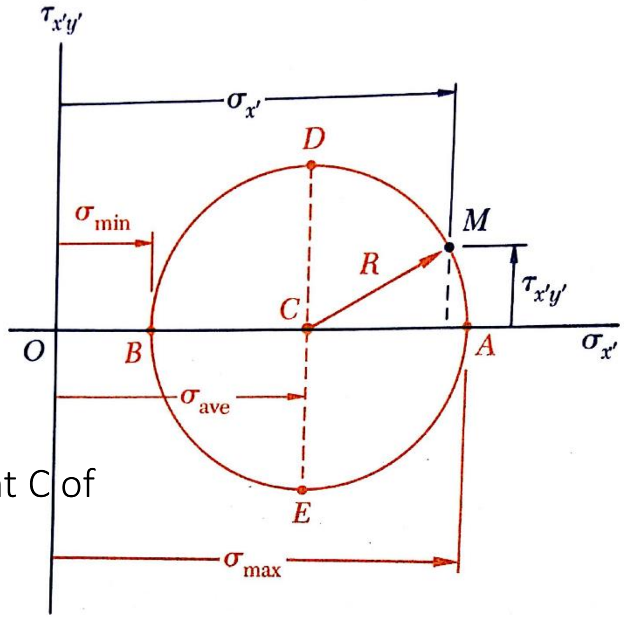
The angle θ in the equations can be eliminated by algebraic simplifications and addition of the two equations:

$$\left(\sigma_x{}' - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}{}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

Where $\frac{\sigma_x+\sigma_y}{2} = \sigma_{ave}$ and $\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}{}^2 = R^2$

So $(\sigma_x{}' - \sigma_{ave})^2 + \tau_{x'y'}{}^2 = R^2$

Which is the equation of a circle of radius R centered at the point C of coordinates $(\sigma_{ave}, 0)$



The point A where the circle intersects the horizontal axis is the maximum value of the normal stress $\sigma_x{}'$ and the other intersection point B is the minimum value. Both points correspond to a zero value of shear stress $\tau_{x'y'}$.
These are the **principle stresses**.

Since $\sigma_{max} = \sigma_{ave} + R$ $\sigma_{min} = \sigma_{ave} - R$

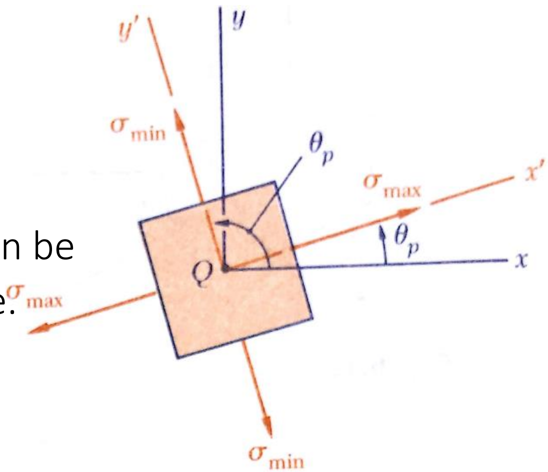
$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The rotation angles that produce the principal stresses with no shear stress is obtained from the equation of shear stress

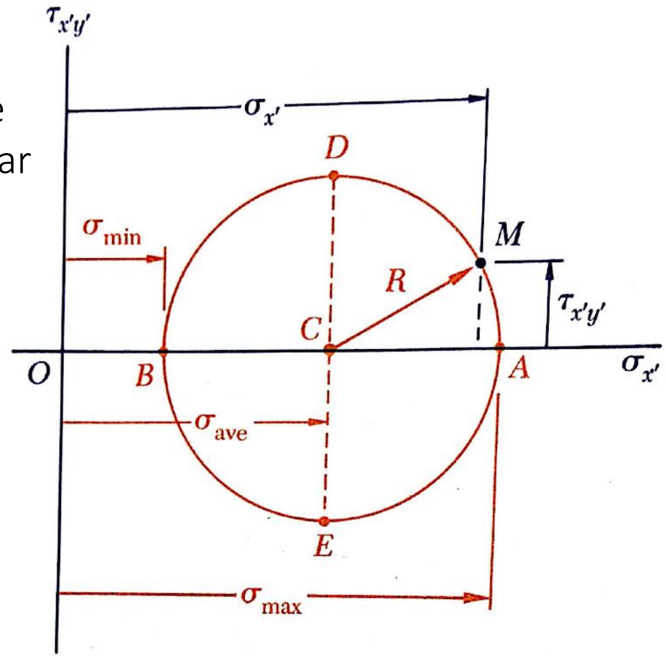
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = 0$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

This equation gives two θ_p values that are 90 apart. Either of them can be used to determine the orientation of the corresponding rotated plane. These planes are the principal planes of stress at point Q



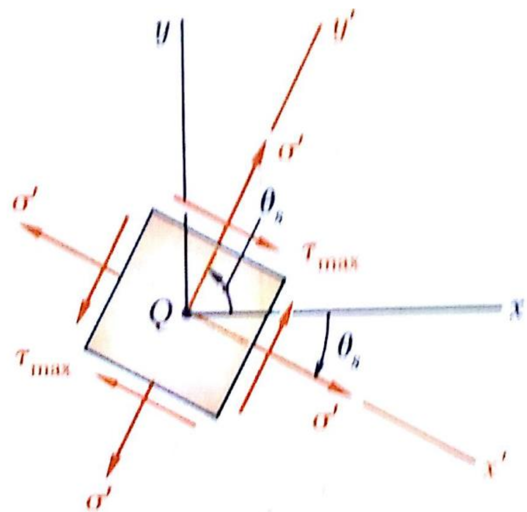
The points D and E are located on the vertical diameter of the circle corresponding to the largest numerical value of the shear stress $\tau_{x'y'}$. These points have the same normal stresses of σ_{ave}. So the rotation that produces the maximum shear stresses can be obtained from the normal stress equations.



$$\sigma_x{}' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= \frac{\sigma_x + \sigma_y}{2}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

This equation gives two θ_s values that are 90 apart. Either of them can be used to determine the orientation of corresponding rotated plane that produces the maximum shear stress which is equal to



$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The normal stress corresponding to the condition of maximum shear stress is

$$\sigma_x{}' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

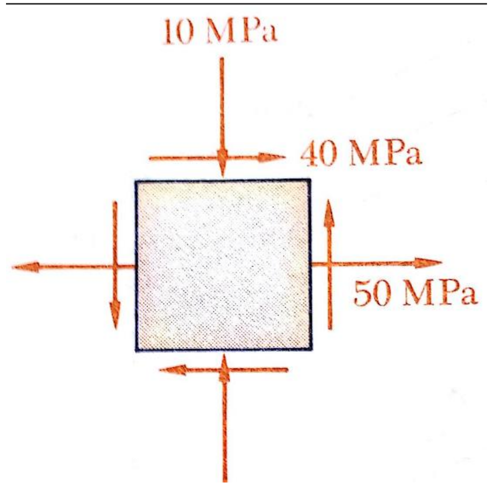
Also

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = -\left(\tan 2\theta_p\right)^{-1} = -\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)^{-1}$$

This means that the angles θ_s and θ_p are 45 apart

So the planes of maximum shear stress are oriented at 45 to the principal planes

Example – Determine the principal planes, principle stresses, maximum shear stress and the corresponding normal stress for the state of plane stress shown



Maximum shear stress criterion (for ductile materials)

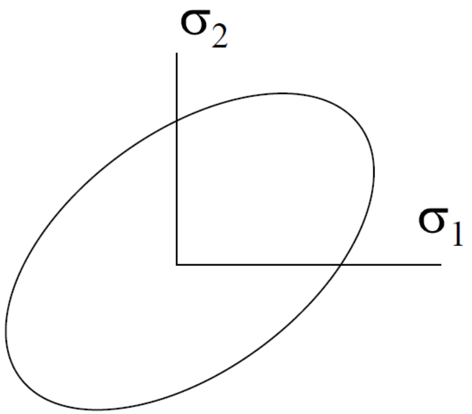
When a ductile material is under uniaxial stress, the value of the normal stress σ_x which will cause the material to yield can be determined simply from a stress-strain diagram obtained by a tensile test.

The material will deform plastically when $\sigma_x > \sigma_{Yield}$

On the other hand when a material is in a state of multiaxial stress, the material may yield when the maximum value of the shear stress exceeds the corresponding value of the shear stress in a tensile-test specimen as it starts to yield.

Maximum shear stress criterion is based on the observation that yield in ductile materials is caused by slippage of the material along oblique surfaces and is due primarily to shear stresses.

In the plane stress condition the material can be represented as a point under principal stresses σ_a, σ_b



Recall that the maximum value of shear stress at a point under a centric axial load is equal to half the value of the corresponding normal axial stress.

Thus at yielding

$$\tau_{max} = \frac{1}{2}\sigma_Y$$

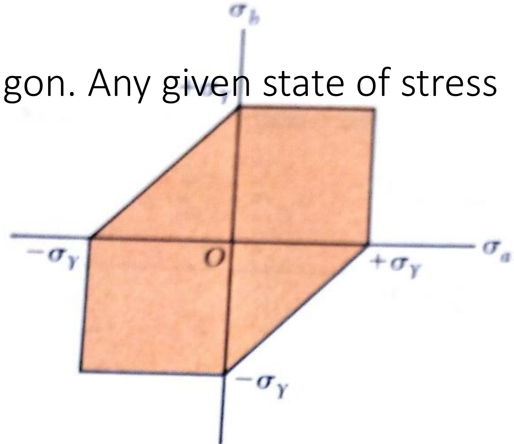
Also for plane stress condition if the principle stresses are both positive or both negative, the maximum value of the shear stress is equal to $\frac{1}{2}|\sigma_{max}|$

Therefore $|\sigma_a| > \sigma_Y$ or $|\sigma_b| > \sigma_Y$

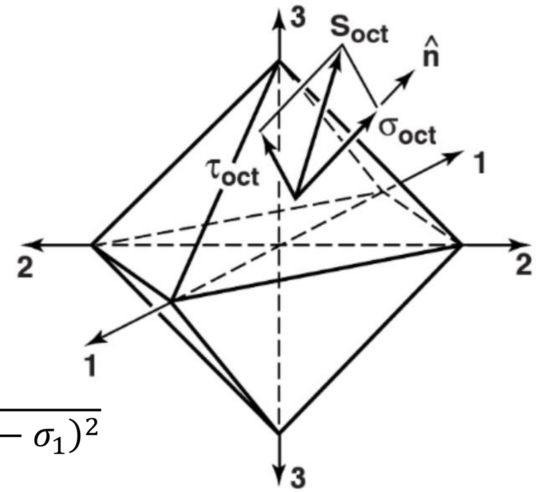
If the maximum stress is positive and the minimum stress negative, the maximum value of the shear stress is equal to $\frac{1}{2}(|\sigma_{max}| - |\sigma_{min}|)$

Therefore $(|\sigma_a| - |\sigma_b|) > \sigma_Y$

These relations produce a hexagon in the xy plane, called Tresca's hexagon. Any given state of stress will be represented in the figure by a point.



Prediction of yielding under multiaxial loading according to the maximum shear stress criterion involves the analysis of the octahedral planes



There are eight octahedral planes making equal angles with the principal stress directions

The shearing stress on these planes is given by

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Or with non-principle stresses:

$$\tau_{oct} = \frac{1}{3}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{yz}{}^2 + \tau_{zx}{}^2 + \tau_{xy}{}^2\right)}$$

The shear strain acting on an octahedral plane is given by

$$\gamma_{oct} = \frac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

Or

$$\gamma_{oct} = \frac{2}{3}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\left(\gamma_{yz}{}^2 + \gamma_{zx}{}^2 + \gamma_{xy}{}^2\right)}$$

Maximum normal stress criterion (for brittle materials)

Brittle materials fail suddenly in a tensile test by rupture without any prior yielding.

When a brittle material is under uniaxial tensile stress, the value of the normal stress which causes it to fail is equal to the ultimate strength of the material as determined from a tensile test.

When a brittle material is under plane stress, the principal stresses are compared to the ultimate strength obtained from the uniaxial tensile test.

Maximum principal stress criterion states that a brittle material will fail when the maximum normal stress exceeds the ultimate strength obtained from the uniaxial tensile test:

$$|\sigma_a| > \sigma_U \text{ or } \qquad |\sigma_b| > \sigma_U$$

This criterion forms a square area centered on the xy plane. The criterion is based on the assumption that the ultimate strength of materials under tension and compression are equal, which is an overestimation for most materials as the presence of cracks and flaws often weaken the material under tension

Maximum distortion energy criterion is based on the determination of the distortion energy in a ductile material, which is the energy consumed by a change in the shape of the material.

Also called von Mises criterion, it states that a material will yield when the maximum value of the distortion energy per unit volume exceeds the distortion energy per unit volume required to cause yield in a tensile test specimen.

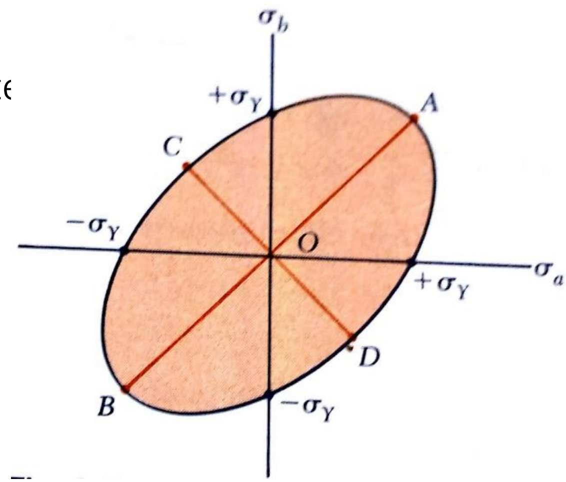
The distortion energy in an isotropic material under plane stress is

$$U_d = \frac{1}{6G}({\sigma_a}^2 - \sigma_a \sigma_b + {\sigma_b}^2)$$

In the case of a tensile test specimen yielding at σ_Y

$$U_Y = \frac{1}{6G}({\sigma_Y}^2)$$

Thus the maximum distortion energy criterion indicates that the mate

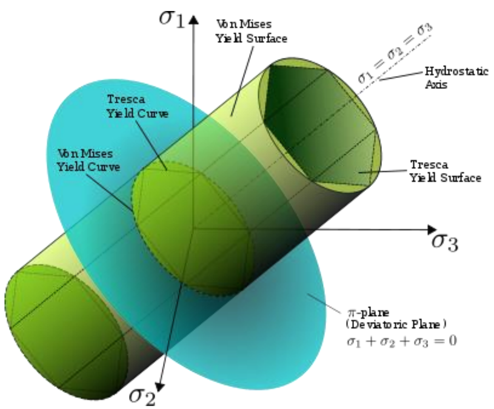


$${\sigma_a}^2 - \sigma_a \sigma_b + {\sigma_b}^2 > {\sigma_Y}^2$$

This equation produces an ellipse in the principal stress plane

It is useful to convert the multiaxial stress state to an equivalent stress

The equivalent or effective stress is the uniaxial stress that is equally distant from the yield surface or located on it



The effective stress or the stress intensity for an elastic material is expressed as

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{yz}{}^2 + \tau_{zx}{}^2 + \tau_{xy}{}^2\right)}$$

And the effective strain as

$$\varepsilon_{eff} = \frac{\sqrt{2}}{2(1+\nu)}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\left(\gamma_{yz}{}^2 + \gamma_{zx}{}^2 + \gamma_{xy}{}^2\right)}$$

And $\sigma_{eff} = E\varepsilon_{eff}$

The von Mises yield criterion is given by

$$\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = 2\sigma_y$$

Or

$$\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left({\tau_{yz}}^2 + {\tau_{zx}}^2 + {\tau_{xy}}^2\right)} = 2\sigma_y$$

In terms of effective stress the criterion is

$$\sigma_{eff} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sigma_y$$

$$\sigma_{eff} = \frac{\sqrt{2}}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left({\tau_{yz}}^2 + {\tau_{zx}}^2 + {\tau_{xy}}^2\right)} = \sigma_y$$

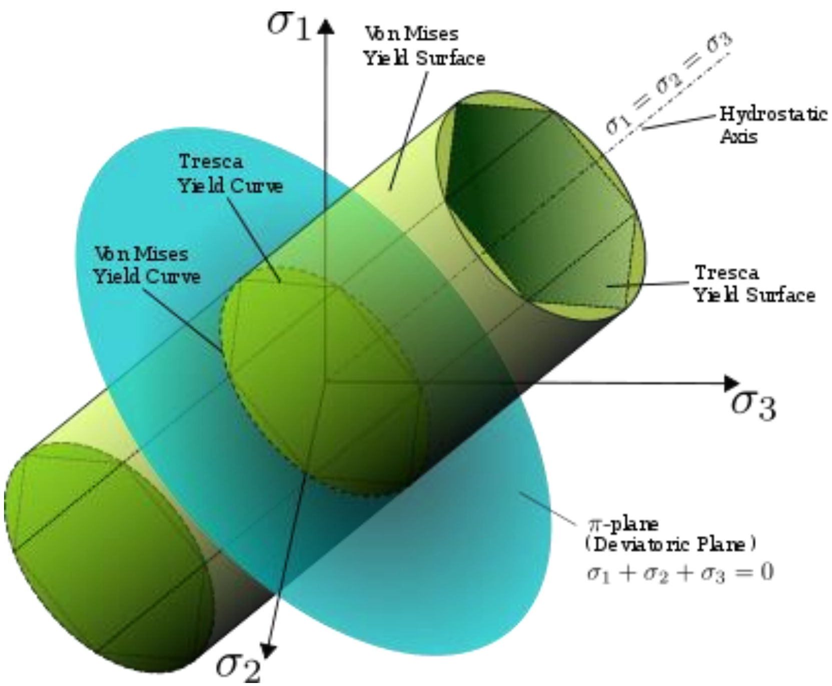
For plane states of stress, the yield condition is the interaction of the cylinder with the principal stress plane, which is a yield ellipse

Notice that the von Mises criterion takes into account the octahedral shear stress

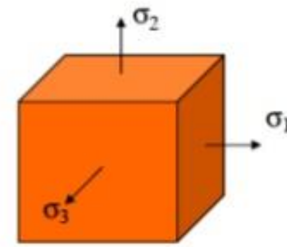
$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left({\tau_{yz}}^2 + {\tau_{zx}}^2 + {\tau_{xy}}^2\right)}$$

The von Mises yield criterion is visualized as a circular cylinder in the stress space



✓ **Body subjected to principal stresses :**



$$U = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$$

$$U=1/2E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

✓ **For the onset of yielding :**

$$Y^2/2E=1/2E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

✓ **Yield function**

$$f = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - Y^2$$

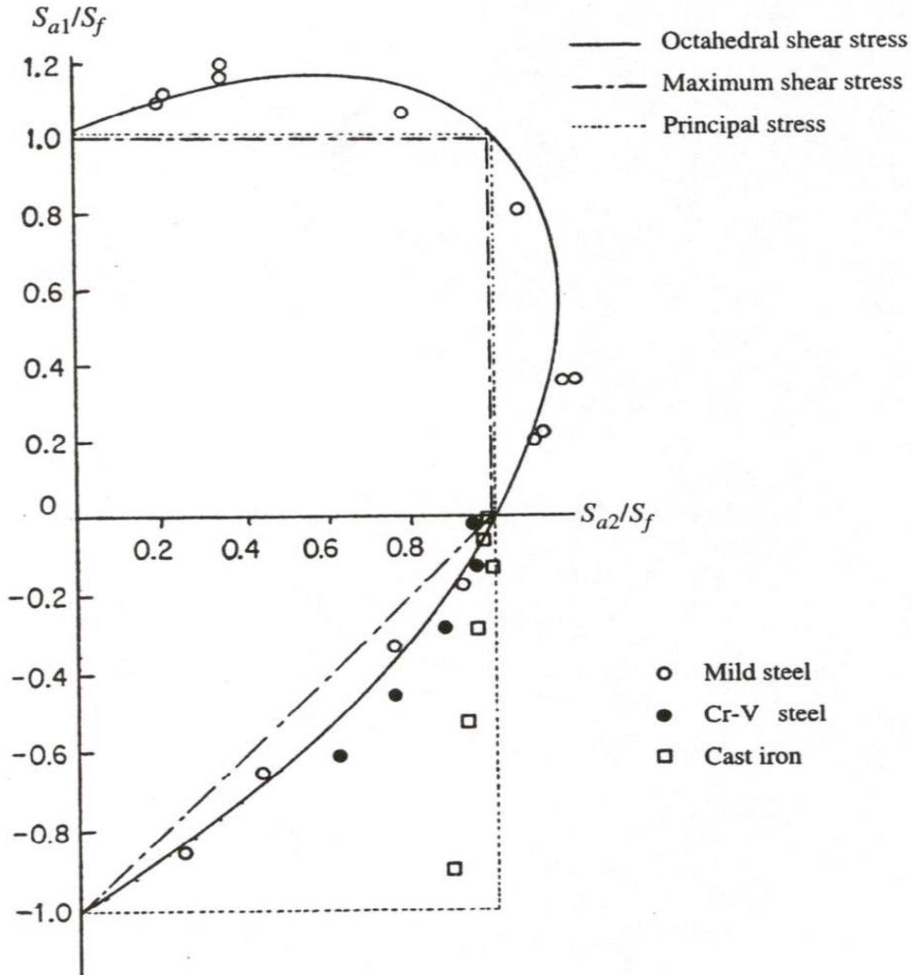
$$f = \sigma_e^2 - Y^2$$

$$\boxed{\text{Yielding} \Rightarrow f = 0, \;\; \text{safe } f < 0}$$

The axis of the cylinder passes through the origin of the coordinates for unyielded material

It is inclined equal amount to the three axes and represents pure hydrostatic stress for elastic deformations.

Yield criteria for deformation of metals under plane stress



The data for the mild steel and Cr-V steel which behave in a ductile manner agree well with the octahedral shear stress (von Mises) criterion

Data for cast iron which behaves in a brittle manner, agrees better with the maximum principal stress criterion:

$$\sigma_1 = \sigma_y$$

Example – Evaluate the yielding stress condition for a ductile cast iron using maximum shear stress, maximum principal stress and maximum distortion energy criteria.

$|\sigma_a| > \sigma_Y$ or $|\sigma_b| > \sigma_Y$ or $(|\sigma_a| - |\sigma_b|) > \sigma_Y$

$|\sigma_a| > \sigma_U$ or $|\sigma_b| > \sigma_U$

$\sigma_a{}^2 - \sigma_a\sigma_b + \sigma_b{}^2 > \sigma_Y{}^2$